

# A Triangle Analog to Pascal's characterizing primes

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**Abstract :** In the conference paper we construct analouge to Pascal's triangle that characterize primes with additional fascinating properties.

## Introduction :

A primality test is an algorithm for determining whether an input number is prime. Amongst other fields of mathematics, it is used for cryptography. Unlike factorization, primality tests do not generally give prime factors, only stating whether the input number is prime or not. Factorization is thought to be a computationally difficult problem, where as primality testing is comparatively easy (its running time is polynomial in the size of the input). A connection between the sequence of prime numbers and the binomial coefficients is given by a well-known characterization of the prime numbers:

*Consider the entries in the  $p$ th row of Pascal's triangle, without the initial and final entries.*

*They are all divisible by  $p$  if and only if  $p$  is a prime. [1]*

In this paper we construct an analouge to pascal's triangle that characterize primes with additional fascinating properties .

We name this triangle **Alkarkhi's triangle** to the memory of Abu Bakr ibn Muhammad ibn al Husayn Al-Karkhi was a 10th-century mathematician and engineer who lived at Baghdad. His three famous surviving works are : Al-Badi' fi'l-hisab, Al-Fakhri fi'l-jabr wa'l-muqabala, and Al-Kafi fi'l-hisab.[2]

**Alkarkhi's triangle** is a geometrical arrangement of numbers defined by the following matrix

$K(i, j) = [a_{ij}]$  such that :

$$a_{ij} = \begin{cases} \binom{i}{j} + (-1)^{j+1} & \text{if } i \geq j > 0 \\ 0 & \text{if } 0 < i < j \end{cases}$$

where  $i$  is the row number and  $j$  is the column number and  $\binom{i}{j}$  is a binomial coefficient .

## Properties of Alkarkhi's Triangle

Alkarkhi's Triangle determines the coefficients which arise in the expansion :

$$\begin{aligned}
\sum_{i=1}^{n-1} K(n, i) x^i &= (1+x)^n - \sum_{i=0}^n (-x)^i \\
&= \sum_{i=0}^n \binom{n}{i} (x)^{n-i} - \sum_{i=0}^n (-x)^i \\
&= \left( \binom{n}{0} (x)^n + \binom{n}{1} (x)^{n-1} + \binom{n}{1} (x)^{n-2} + \dots + \binom{n}{n-1} (x)^1 + \binom{n}{n} (x)^0 \right) \\
&\quad - (1 - x + x^2 - \dots + (-1)^n x^n) \\
&= \left( \binom{n}{1} - 1 \right) (x)^{n-1} + \left( \binom{n}{2} + 1 \right) (x)^{n-2} + \dots + \left( \binom{n}{n-1} + (-1)^{n-1} \right) (x)^1
\end{aligned}$$

**Lemma 1:** The enteries of the  $i^{th}$  row represents the coefficients of the above expansion of the generating function and all enteries can be constructed from the recurrence formula :

$$K(n+1, d) + (-1)^d = K(n, d) + k(n, d-1)$$

**Proof:** using the identity

$$\binom{n}{d-1} + \binom{n}{d} = \binom{n+1}{d}$$

and the definition

$$K(n, d) = \binom{n}{d} + (-1)^{d+1}$$

with direct substitution

$$K(n, d-1) + (-1)^{d-1} + K(n, d) + (-1)^d = K(n+1, d) + (-1)^d$$

and consequently

$$K(n+1, d) + (-1)^d = K(n, d) + k(n, d-1)$$

The first few enteries of **Alkarkhi's Triangle** are given below :

$$K(n, d) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 9 & 11 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 14 & 21 & 14 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 20 & 36 & 34 & 22 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 27 & 57 & 69 & 57 & 27 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 35 & 85 & 125 & 127 & 83 & 37 & 8 & 2 & 0 & 0 & 0 & 0 \\ 11 & 44 & 121 & 209 & 253 & 209 & 121 & 44 & 11 & 0 & 0 & 0 & 0 \\ 12 & 54 & 166 & 329 & 463 & 461 & 331 & 164 & 56 & 10 & 2 & 0 & 0 \\ 13 & 65 & 221 & 494 & 793 & 923 & 793 & 494 & 221 & 65 & 13 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

### Alkarkhi's Triangle characterizes prime numbers

we present the following amazing property of **Alkarkhi's Triangle** to characterize primes

**Theorem 1** :  $p$  is prime if and only if all the entires of the  $(p-1)^{th}$  row in **Alkarkhi's Triangle** are zero mod  $p$

Proof : Obviously each entery of the  $(p-1)^{th}$  row in Alkarkhi's Triangle has the form

$$\binom{p-1}{d} + (-1)^{d+1}$$

and using :

For any  $0 \leq d \leq p-1$  :  $\binom{p-1}{d} + (-1)^{d+1} \equiv 0 \pmod{p}$  if and only if  $p$  is prime .

Notice that all the enteries in  $(p-1)^{th}$  row in bold divisible by the number  $p$  indicates the primality of  $p$

2 Prime	<b>2</b>	0	0	0	0	0	0	0	0	0	0	0	0
3 Prime	<b>3</b>	0	0	0	0	0	0	0	0	0	0	0	0
4 Not Prime	4	2	2	0	0	0	0	0	0	0	0	0	0
5 Prime	<b>5</b>	<b>5</b>	<b>5</b>	0	0	0	0	0	0	0	0	0	0
6 Not Prime	6	9	11	4	2	0	0	0	0	0	0	0	0
7 Prime	<b>7</b>	<b>14</b>	<b>21</b>	<b>14</b>	<b>7</b>	0	0	0	0	0	0	0	0
8 Not Prime	8	20	36	34	22	6	2	0	0	0	0	0	0
9 Not Prime	9	27	57	69	57	27	9	0	0	0	0	0	0
10 Not Prime	10	35	85	125	127	83	37	8	2	0	0	0	0
11 Prime	<b>11</b>	<b>44</b>	<b>121</b>	<b>209</b>	<b>253</b>	<b>209</b>	<b>121</b>	<b>44</b>	<b>11</b>	0	0	0	0
12 Not Prime	12	54	166	329	463	461	331	164	56	10	2	0	0
13 Prime	<b>13</b>	<b>65</b>	<b>221</b>	<b>494</b>	<b>793</b>	<b>923</b>	<b>793</b>	<b>494</b>	<b>221</b>	<b>65</b>	<b>13</b>	0	0
*****	*	*	*	*	*	*	*	*	*	*	*	*	*

Another property of **Alkarkhi's Triangle** to charecterize primality of a given number .

**Theorem 2** :  $p$  is prime if and only if all the entires of the  $p^{th}$  row not exceeding the enteries of the diagonal are  $(1, -1, 1, -1, 1, \dots)$  mod  $p$  .

Proof : A direct consequence of the construction and the recurrence formula ,

Similarly as in the previous theorem each entery of the  $(p)^{th}$  row in Alkarkhi's Triangle has the form

$$\binom{p}{d} + (-1)^{d+1}$$

in  $p^{th}$  row

$$\begin{aligned} & \left[ \binom{p}{1} + (-1)^{1+1}, \binom{p}{2} + (-1)^{2+1}, \binom{p}{3} + (-1)^{3+1}, \binom{p}{4} + (-1)^{4+1}, \dots \right] \\ & \equiv \left[ 0 \pmod{p} + (-1)^{1+1}, 0 \pmod{p} + (-1)^{2+1}, 0 \pmod{p} + (-1)^{3+1}, 0 \pmod{p} + (-1)^{4+1}, \dots \right] \end{aligned}$$

$$\begin{aligned} &\equiv [ (-1)^{1+1}, (-1)^{2+1}, (-1)^{3+1}, (-1)^{4+1}, (-1)^{5+1}, \dots ] \bmod p \\ &\quad [ 1, -1, 1, -1, 1, \dots ] \bmod p \end{aligned}$$

#### Alkakhi's triangle rows sum

**Lemma 2 :** The sum for the  $n^{th}$  row gives power of 2 as

$$\sum_{d=1}^n K(n, d) = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 2^n - 1 & \text{if } n \text{ is even} \end{cases}$$

Proof :

$$\begin{aligned} \sum_{d=1}^n K(n, d) &= \sum_{d=1}^n \left\{ \binom{n}{d} + (-1)^{d+1} \right\} \\ &= \sum_{d=1}^n \binom{n}{d} + \sum_{d=1}^n (-1)^{d+1} \\ &= (2^n - 1) + \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 2^n - 1 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Example: The sum of the  $7^{th}$  row enteries :

$$8 \quad 20 \quad 36 \quad 34 \quad 22 \quad 6 \quad 2$$

is given as  $8 + 20 + 36 + 34 + 22 + 6 + 2 = 128 = 2^7$

Example : The sum of the  $10^{th}$  row enteries

$$11 \quad 44 \quad 121 \quad 209 \quad 253 \quad 209 \quad 121 \quad 44 \quad 11$$

is given by  $11 + 44 + 121 + 209 + 253 + 209 + 121 + 44 + 11 = 1023 = 2^{10} - 1$

#### Alkakhi's Triangle rows alternating sign sums

**Lemma 3 :** The alternating sign sum for the  $n^{th}$  row of Alkarkhi's Triangle is given by :

$$\sum_{d=1}^n (-1)^{d+1} K(n, d) = n + 1$$

Proof

$$\begin{aligned} \sum_{d=0}^n (-1)^d \binom{n}{d} &= 0 \\ \Rightarrow 1 + \sum_{d=1}^n (-1)^d \binom{n}{d} &= 1 + \sum_{d=1}^n (-1)^d \{ K(n, d) + (-1)^d \} \\ &= 1 + \sum_{d=1}^n (-1)^d \{ K(n, d) + (-1)^d \} = 1 + \sum_{d=1}^n \{ (-1)^d K(n, d) + 1 \} \\ &= \sum_{d=1}^n (-1)^d K(n, d) + n + 1 = 0 \\ \Rightarrow \sum_{d=1}^n (-1)^{d+1} K(n, d) &= n + 1 \end{aligned}$$

Example : The alternating sign sum of the  $10^{th}$  row enteries

$$11 \quad 44 \quad 121 \quad 209 \quad 253 \quad 209 \quad 121 \quad 44 \quad 11 \quad 0 \quad 0 \quad 0 \quad 0$$

is given by  $11 - 44 + 121 - 209 + 253 - 209 + 121 - 44 + 11 = 10 + 1$

### Alkaraji's Triangle rising diagonals and Fibonacci numbers:

Using the well known identity

$$F_{n+1} = \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-d}{d}$$

we noticed that Fibonacci numbers are located in the rising diagonals of Alkarkhi's triangle

$$K(n, d) = \binom{n}{d} + (-1)^{d+1}$$

$$\begin{aligned} F_{n+1} &= 1 + \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-d}{d} \\ &= 1 + \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} (K(n-d, d) + (-1)^d) \\ &= 1 + \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} K(n-d, d) + \begin{cases} -1 & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is odd} \\ 0 & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is even} \end{cases} \\ &= \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} K(n-d, d) + \begin{cases} 0 & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is odd} \\ 1 & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is even} \end{cases} \end{aligned}$$

finaly

$$F_{n+1} = \begin{cases} \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} K(n-d, d) & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is odd} \\ \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} K(n-d, d) + 1 & \text{if } \lfloor \frac{n}{2} \rfloor \text{ is even} \end{cases}$$

Example : To compute  $F_8$ , observe that  $n=9$  and  $\lfloor \frac{9}{2} \rfloor = 4$  is even, we apply the formula :

$$F_8 = \sum_{d=1}^4 K(9-d, d) + 1 = K(8, 1) + K(7, 2) + K(6, 3) + K(5, 4) + 1 = 8 + 7 + 6 + 5 + 1 = 27$$

Or one can compare with rising diagonals in Alkarkhi's triangle

2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0
5	5	5+	↗	0	0	0	0
6	9+	↗	11	4	2	0	0
7+	↗	14	21	14	7	0	0
8	20	36	34	22	6	2	0
9	27	57	69	57	27	9	0

**References :**

- [1] Rashed, Roshdi (1970–80). "Al-KarajI (or Al-Karkhī), Abu Bakr Ibn Muhammad Ibn al Husayn". Dictionary of Scientific Biography. NewYork:Charles Scribner's Sons. ISBN 978-0-684-10114-9.
- [2] Karl Dilcher and Kenneth B.Stolasky. Apascal-Type Triangle Characterizing Twin Primes.Amer.Math.Monthly,112(8):673-681,2005.